Lecture Note 1: Modular Representations and Brauer Characters of A_5

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Overview:
In this lecture, we explore the modular representation theory of the alternating group A_5, with a focus on the computation of Brauer characters over fields of positive characteristic, congruence conditions, and the interplay between ordinary and modular character tables. We also sketch proofs and highlight computational techniques grounded in classical and modern algebraic tools.

1. Modular Representation Framework

We begin with a classical example: the group A_5, which is simple and of order 60. The goal is to study its representations in characteristic p, focusing on $p = 2$.

Let us recall:

- A_5 has five irreducible ordinary representations over $\mathbb{C}$, of degrees $1, 3, 3', 4, 5$.
- We aim to understand its modular representations over $\mathbb{F}_2$ and its algebraic closure $\overline{\mathbb{F}}_2$.

2. Rational and Congruence Conditions

The character values lie in cyclotomic fields; for instance:

- Roots of unity (e.g., $\epsilon = e^{2\pi i/5}$) describe irreducible character values.
- Fields such as $\mathbb{Q}(\sqrt{5})$ are naturally involved in the field of definition.

Congruence constraints arise:

- Modulo 2, 3, 5 conditions must be satisfied due to integrality and class function congruences.
- Algebraic integers such as $\frac{-1\pm\sqrt{5}}{2}$ serve as trace values.

3. Brauer Characters and Decomposition

When working modulo p, ordinary characters are projected to Brauer characters by:

- Removing columns corresponding to p-singular classes.
- Working only with p-regular classes to construct modular representations.

We derive that:

$$A_5 \text{ has a two-dimensional irreducible representation over } \overline{\mathbb{F}}_2$$

4. Trace Fields and Frobenius Automorphism

From character values:

- The trace field of the representation lies in $\mathbb{F}_4$.
- Application of the Frobenius automorphism $x \mapsto x^2$ yields conjugate representations.

This is grounded in the result that:
Over finite fields, irreducibility is reflected in the trace field via the Frobenius action.

5. Decomposition Matrix Construction

Decomposition matrix: records how ordinary irreducible characters decompose into irreducible Brauer characters.

- Rows: Brauer irreducibles
- Columns: Ordinary irreducibles
- Entries: Non-negative integers indicating multiplicities

This matrix enables:

- Recovery of Brauer characters from known ordinary tables.
- Efficient computation using GAP and nullspace analysis.

6. Theorem on Irreducibility via GCD Condition

Theorem (Irreducibility Criterion):
Let G be a finite group, $\chi \in \mathrm{Irr}(G)$, and p a prime. If:

$$\gcd\left(\frac{|G|}{\chi(1)}, p\right) = 1$$

then the modular reduction of the representation associated with χ is irreducible over $\overline{\mathbb{F}}_p$.

Sketch: This follows by observing that the modular reduction of an integral representation preserves irreducibility under the stated GCD condition.

7. Frobenius–Schur Indicator

To determine real, complex, or quaternionic types:

$$\sum_{g \in G} \chi(g^2)$$

In matrix form, it involves computing:

$$\sum_{g \in G} x_{ij}(g) x_{kl}(g^{-1})$$

which gives:

$$\frac{|G|}{\chi(1)} \delta_{jk} \delta_{il}$$

This relates to the size of the representation matrices and provides structural insight.

Conclusion

We have used character-theoretic tools, congruence conditions, and computational methods to identify modular irreducibles for A_5. The decomposition matrix serves as a compact encoding of the modular theory, bridging between classical and modern algebraic insights.